Warsaw University of Technology

Faculty of Power and Aeronautical Engineering

WARSAW UNIVERSITY OF TECHNOLOGY

Institute of Aeronautics and Applied Mechanics

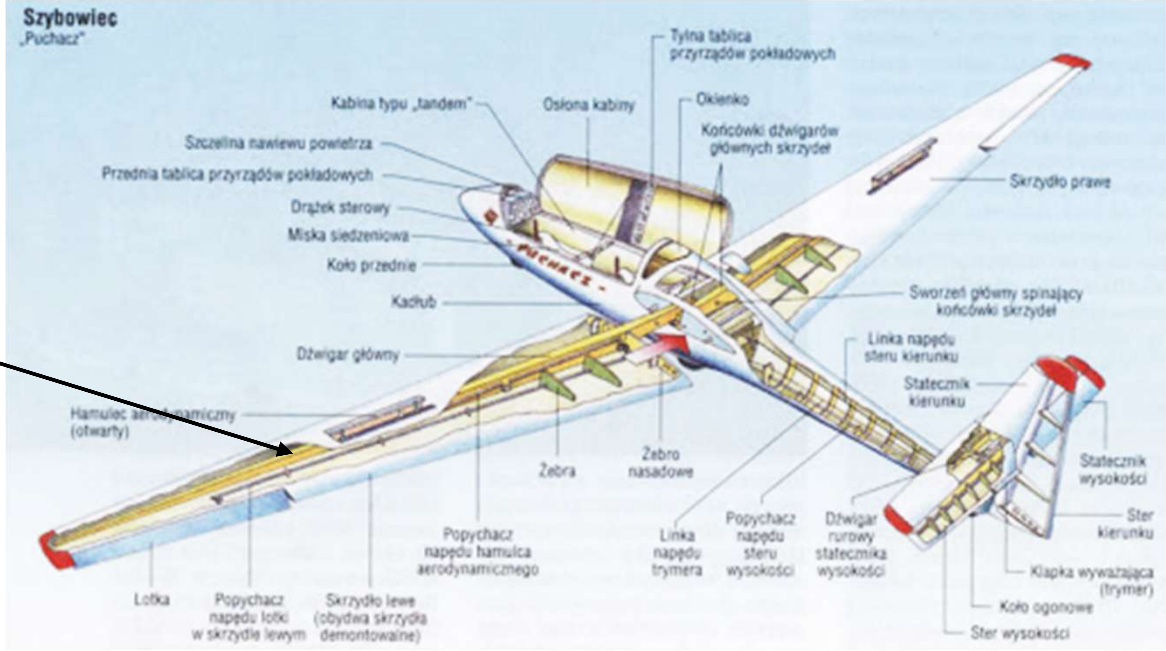
Finite element method (FEM)

1D beam finite element

05.2021

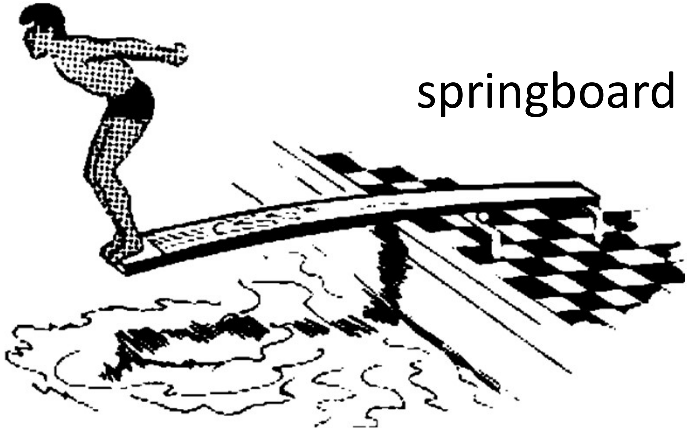
Examples of beams

wing spar



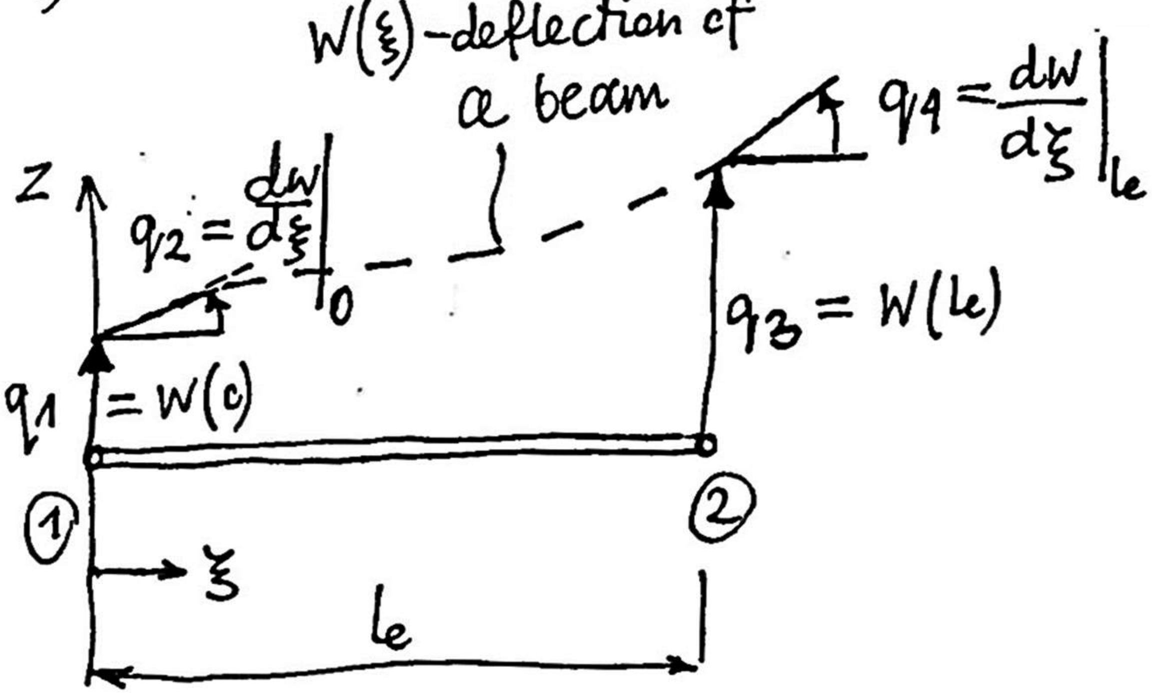
footbridge

springboard



A BEAM FINITE ELEMENT

(bending in one plane)



 q_1, q_3 - translational degrees of freedom (mm)

 q_2, q_4 - rotational degrees of freedom.

$n = 2, \quad n_p = 2, \quad n_e = 2 \cdot 2 = 4$

4 nodal parameters ⟶ 4 constants α_i in $w(\xi)$

$$w(\xi) = \alpha_1 + \alpha_2 \cdot \xi + \alpha_3 \cdot \xi^2 + \alpha_4 \cdot \xi^3 = \lfloor 1, \xi, \xi^2, \xi^3 \rfloor \cdot \begin{Bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \end{Bmatrix}$$

$$\frac{dw}{d\xi} = 0 + \alpha_2 + 2\alpha_3 \cdot \xi + 3\alpha_4 \cdot \xi^2$$

$$w(0) = q_1 = 1 \cdot \alpha_1 + 0 \cdot \alpha_2 + 0 \cdot \alpha_3 + 0 \cdot \alpha_4$$

$$\left.\frac{dw}{d\xi}\right|_0 = q_2 = 0 \cdot \alpha_1 + 1 \cdot \alpha_2 + 2 \cdot 0 \cdot \alpha_3 + 3 \cdot 0^2 \cdot \alpha_4$$

$$w(l_e) = q_3 = 1 \cdot \alpha_1 + l_e \cdot \alpha_2 + l_e^2 \cdot \alpha_3 + l_e^3 \cdot \alpha_4$$

$$\left.\frac{dw}{d\xi}\right|_{l_e} = q_4 = 0 \cdot \alpha_1 + 1 \cdot \alpha_2 + 2 l_e \cdot \alpha_3 + 3 l_e^2 \cdot \alpha_4$$

$$\begin{Bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{Bmatrix}_e = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & l_e & l_e^2 & l_e^3 \\ 0 & 1 & 2l_e & 3l_e^2 \end{bmatrix} \cdot \begin{Bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \end{Bmatrix}$$

$$\begin{Bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & l_e & l_e^2 & l_e^3 \\ 0 & 1 & 2l_e & 3l_e^2 \end{bmatrix}^{-1} \cdot \begin{Bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{Bmatrix}_e$$

$$W(\xi) = \lfloor 1, \xi, \xi^2, \xi^3 \rfloor \cdot \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -\frac{3}{l_e^2} & -\frac{2}{l_e} & \frac{3}{l_e^2} & -\frac{1}{l_e} \\ \frac{2}{l_e^3} & \frac{1}{l_e^2} & -\frac{2}{l_e^3} & \frac{1}{l_e^2} \end{bmatrix} \cdot \begin{Bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{Bmatrix}_e =$$

$$= \lfloor N_1(\xi), N_2(\xi), N_3(\xi), N_4(\xi) \rfloor \cdot \underset{4\times1}{\{q\}_e} = \underset{1\times4}{\lfloor N(\xi) \rfloor} \cdot \underset{4\times1}{\{q\}_e}$$

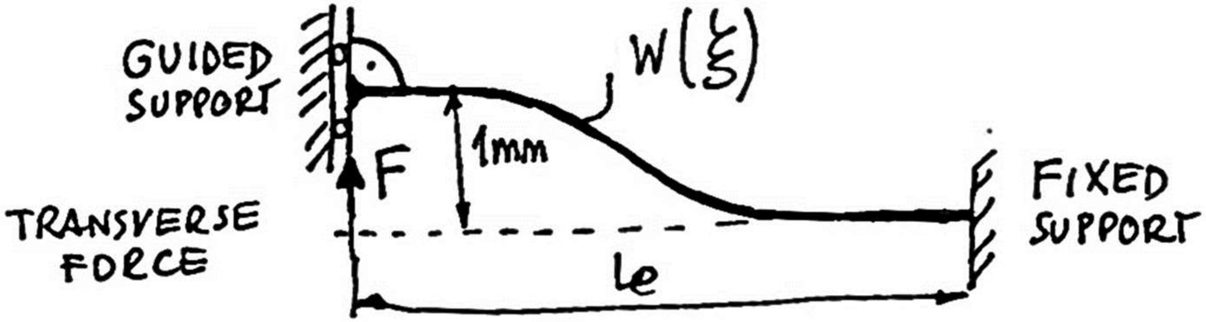
where:

$$\left.\begin{aligned} N_1(\xi) &= 1 - \frac{3}{l_e^2}\xi^2 + \frac{2}{l_e^3}\xi^3 \\ N_2(\xi) &= \xi - \frac{2}{l_e}\cdot\xi^2 + \frac{1}{l_e^2}\cdot\xi^3 \\ N_3(\xi) &= \frac{3}{l_e^2}\xi^2 - \frac{2}{l_e^3}\xi^3 \\ N_4(\xi) &= -\frac{1}{l_e}\xi^2 + \frac{1}{l_e^2}\xi^3 \end{aligned}\right\} \text{shape functions of a beam F.E.}$$

1°) $\lfloor q \rfloor_e = \lfloor 1\,\mathrm{mm}\,, 0\,, 0\,, 0 \rfloor$

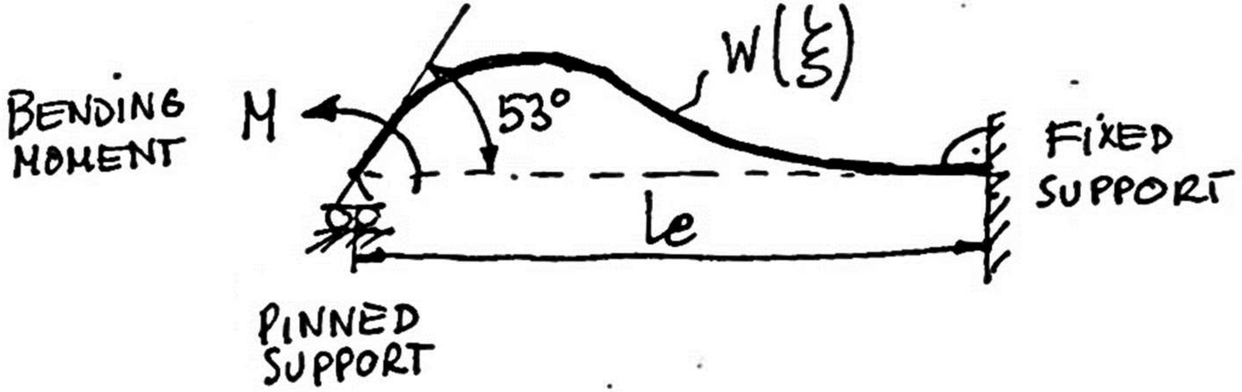
$$w(\xi) = N_1(\xi)\cdot 1\,\mathrm{mm} + N_2(\xi)\cdot 0 + N_3(\xi)\cdot 0 + N_4(\xi)\cdot 0 =$$

$$= N_1(\xi)\cdot 1\,\mathrm{mm}$$

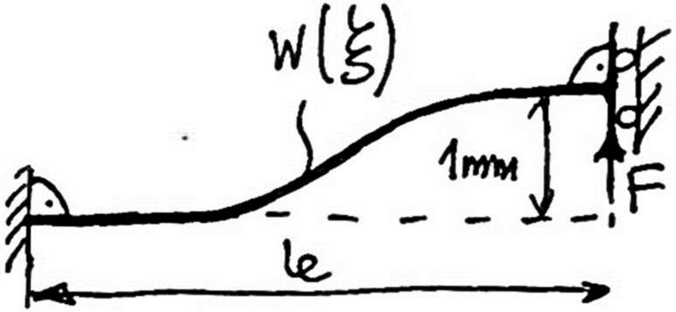


$$2^{\circ})\quad \lfloor q \rfloor_e = \lfloor 0,\ 1,\ 0,\ 0 \rfloor \quad \Rightarrow \quad w(\xi) = N_2(\xi)$$

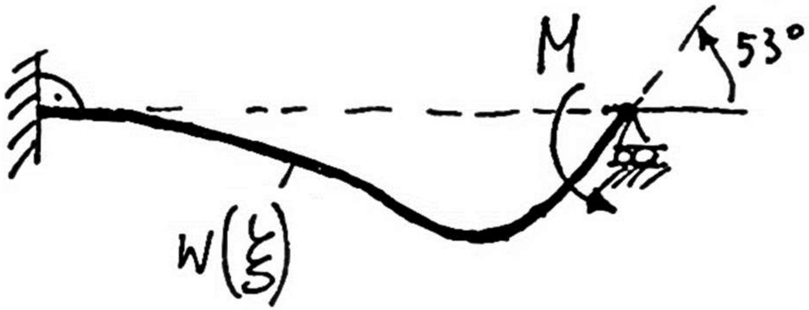
$$1\,\mathrm{rad} = \left(\frac{180}{\pi}\right)^{\circ} \approx 53^{\circ}$$



3°) $\lfloor q \rfloor_e = \lfloor 0,\ 0,\ 1\,\text{mm},\ 0 \rfloor \Rightarrow w(\xi) = N_3(\xi)\cdot 1\,\text{mm}$



4°) $\lfloor q \rfloor_e = \lfloor 0,\ 0,\ 0,\ 1 \rfloor \Rightarrow w(\xi) = N_4(\xi)$



$$N' = \frac{dN}{d\xi} \quad , \quad N'' = \frac{d^2N}{d\xi^2} \quad , \quad N''' = \frac{d^3N}{d\xi^3}$$

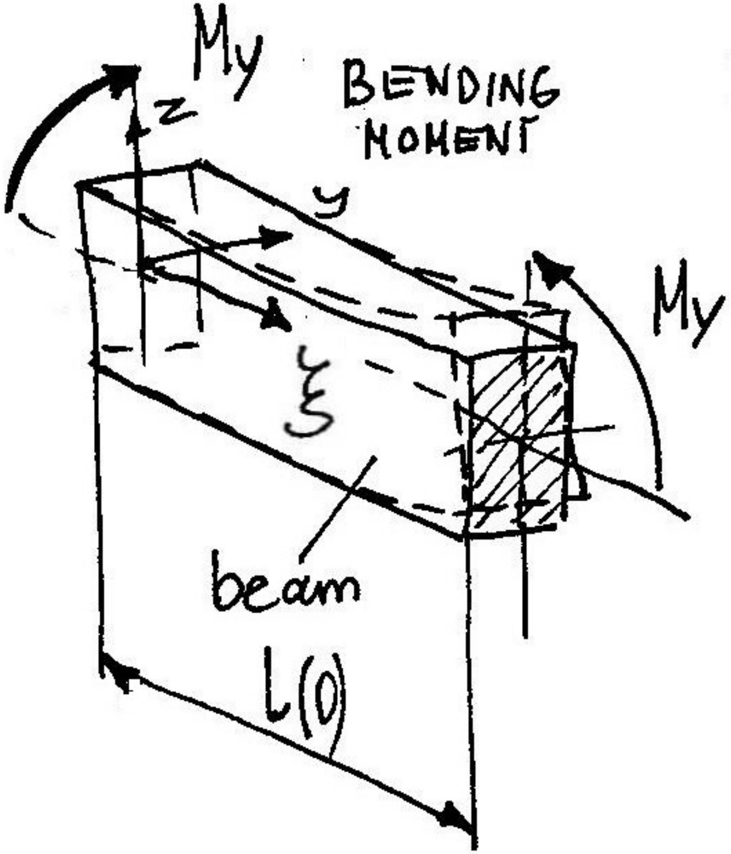
$$N_1' = \underbrace{-\frac{6}{l_e^2}\xi + \frac{6}{l_e^3}\xi^2}_{\text{quadratic function}} \quad , \quad N_1'' = \underbrace{-\frac{6}{l_e^2} + \frac{12}{l_e^3}\xi}_{\text{linear function}} \quad , \quad N_1''' = \underbrace{\frac{12}{l_e^3}}_{\text{constant value}}$$

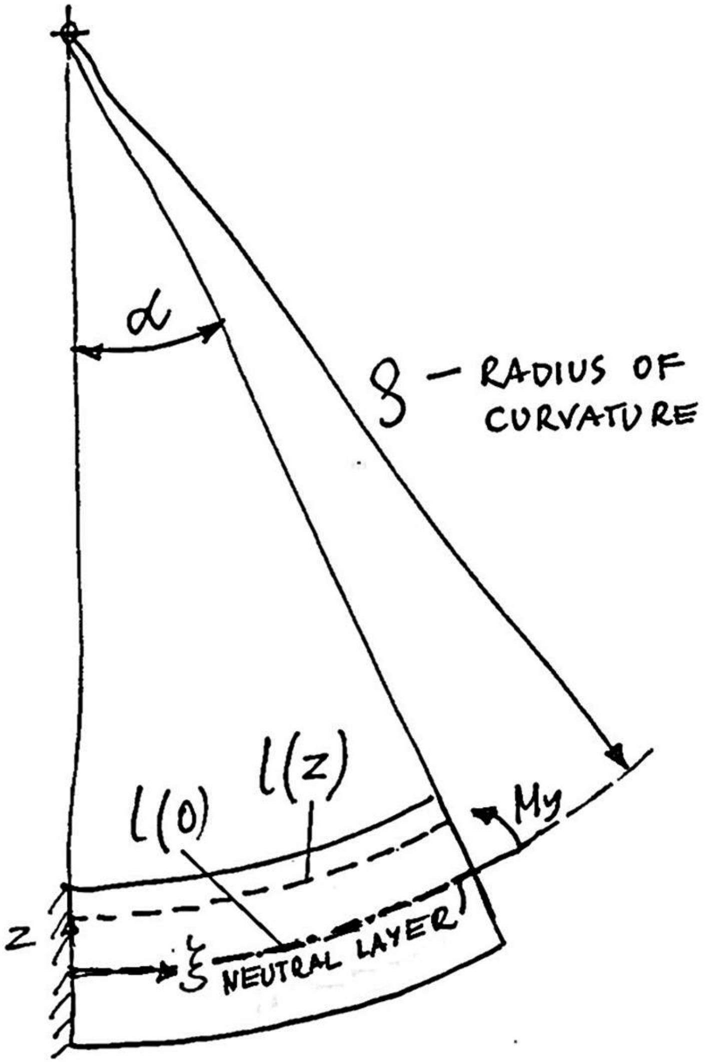
$$N_2' = 1 - \frac{4}{l_e}\xi + \frac{3}{l_e^2}\xi^2 \quad , \quad N_2'' = -\frac{4}{l_e} + \frac{6}{l_e^2}\xi \quad , \quad N_2''' = \frac{6}{l_e^2}$$

$$N_3' = \frac{6}{l_e^2}\xi - \frac{6}{l_e^3}\xi^2 \quad , \quad N_3'' = \frac{6}{l_e^2} - \frac{12}{l_e^3}\xi \quad , \quad N_3''' = -\frac{12}{l_e^3}$$

$$N_4' = -\frac{2}{l_e}\xi + \frac{3}{l_e^2}\xi^2 \quad , \quad N_4'' = -\frac{2}{l_e} + \frac{6}{l_e^2}\xi \quad , \quad N_4''' = \frac{6}{l_e^2}$$

BENDING WITHOUT SHEAR FORCE (PURE BENDING)

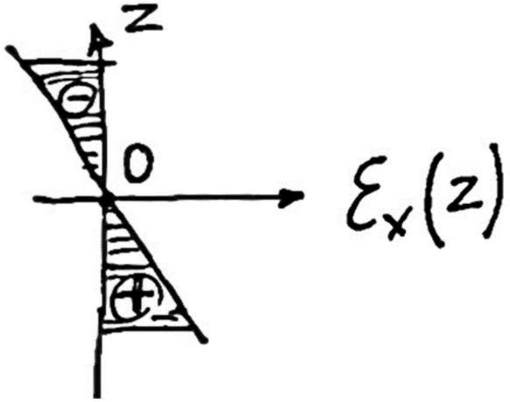




$$\varepsilon_x(z) = \frac{l(z)-l(0)}{l(0)} = \frac{\alpha(\rho-z)-\alpha\rho}{\alpha\rho} = -\frac{z}{\rho}$$

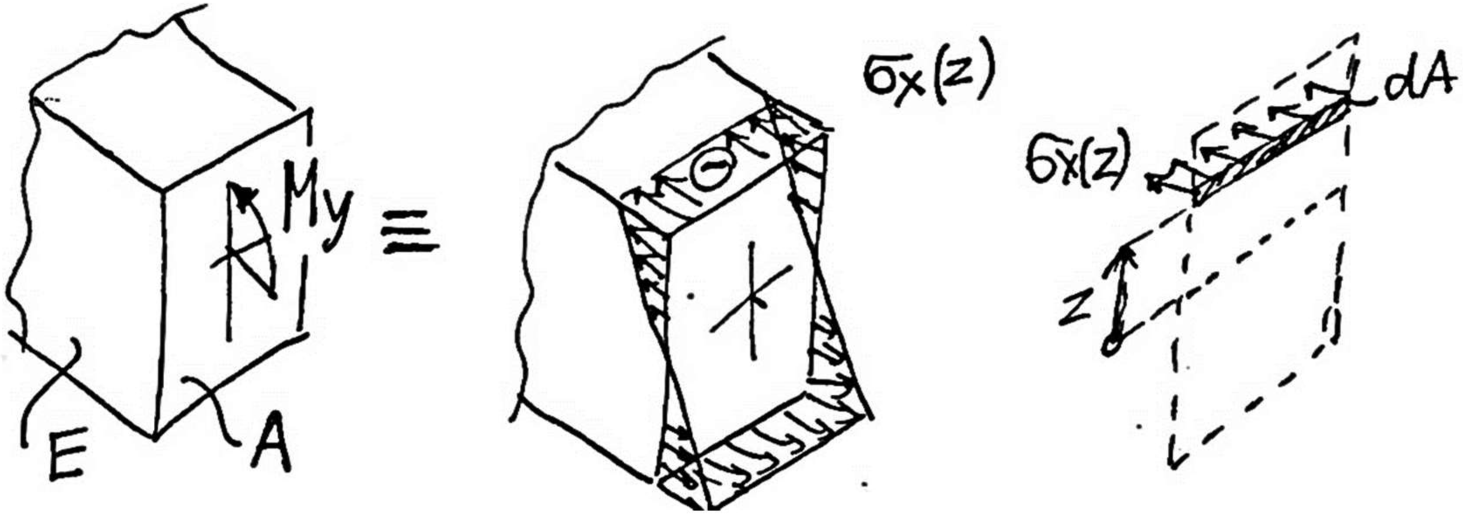
CURVATURE $\mathcal{H} = \frac{1}{\rho} \cong \frac{d^2w}{d\xi^2} = w''$

STRAIN $\varepsilon_x(z) = -z \cdot w''$



STRESS

$$\sigma_x(z) = E \cdot \varepsilon_x(z) = -Ez \cdot w''$$

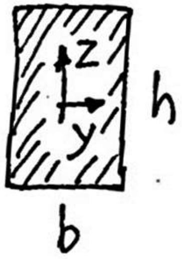


$$M_y = -\int_A \sigma_x(z)\cdot z\,dA = -\int_A -Ezw''\cdot z\,dA =$$

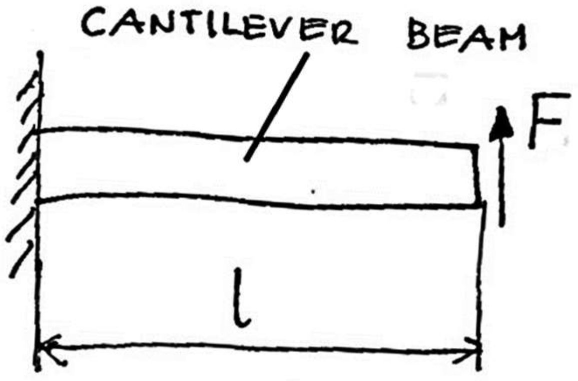
$$= Ew''\underbrace{\int_A z^2 dA}_{\text{Second moment of area } J_y} \Rightarrow \boxed{M_y = EJ_y w''}$$

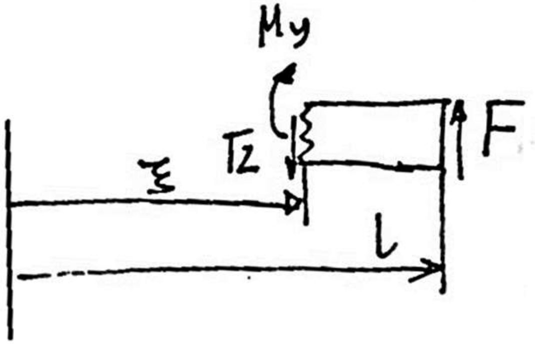
BENDING MOMENT IN A BEAM

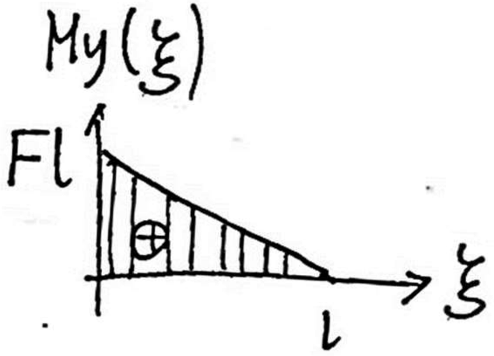
$$J_y \doteq \int_A z^2 dA = \frac{bh^3}{12}$$ for a rectangle



BENDING WITH SHEAR FORCE







$$T_z = F$$

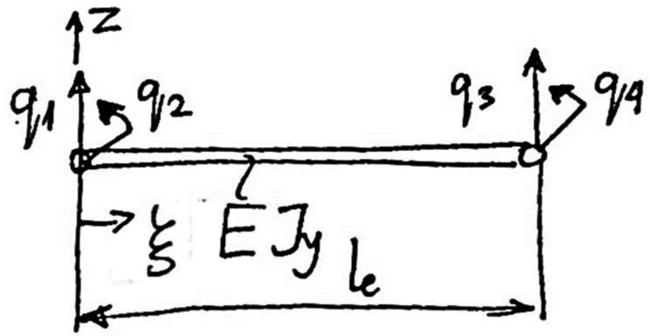
$$M_y - F(l-\xi) = 0$$

$$M_y = T_z(l-\xi)$$

$$\frac{dM_y}{d\xi} = -T_z$$

SHEAR FORCE
IN A BEAM $\boxed{T_z = -EJ_y w'''}$

BEAM FINITE ELEMENT



$$\underset{1\times4}{\lfloor q \rfloor_e} = \lfloor q_1, q_2, q_3, q_4 \rfloor$$

deflection: $w(\xi) = \underset{1\times4}{\lfloor N \rfloor} \cdot \underset{4\times1}{\{q\}_e}$ – polynomial of the 3rd order

bending moment: $M_y(\xi) = EJ_y\, w'' = EJ_y \underset{1\times4}{\lfloor N'' \rfloor} \cdot \underset{4\times1}{\{q\}_e}$ – linear function

shear force: $T_z(\xi) = -EJ_y\, w''' = -EJ_y \underset{1\times4}{\lfloor N''' \rfloor} \cdot \underset{4\times1}{\{q\}_e}$ – constant value

elastic strain energy (pure bending)

$$U_e = \int\limits_{\Omega_e} \frac{1}{2}\sigma_x \varepsilon_x \, d\Omega_e = \int\limits_{\Omega_e} \frac{1}{2}\left(-EZ\cdot W''\right)\cdot\left(-zW''\right) d\Omega_e =$$

$$= \int\limits_{\Omega_e} \frac{1}{2} E W'' W'' z^2 d\Omega_e = \int\limits_{0}^{l_e} \frac{1}{2} E W'' W'' \cdot \int\limits_{A} z^2 dA \, d\xi =$$

$$= \int\limits_{0}^{l_e} \frac{EJ_y}{2} \, W''\cdot W'' d\xi = \int\limits_{0}^{l_e} \frac{EJ_y}{2} \underset{1\times4}{\lfloor q \rfloor_e} \cdot \underset{4\times1}{\{N''\}} \cdot \underset{1\times4}{\lfloor N'' \rfloor} \cdot \underset{4\times1}{\{q\}_e} \, d\xi$$

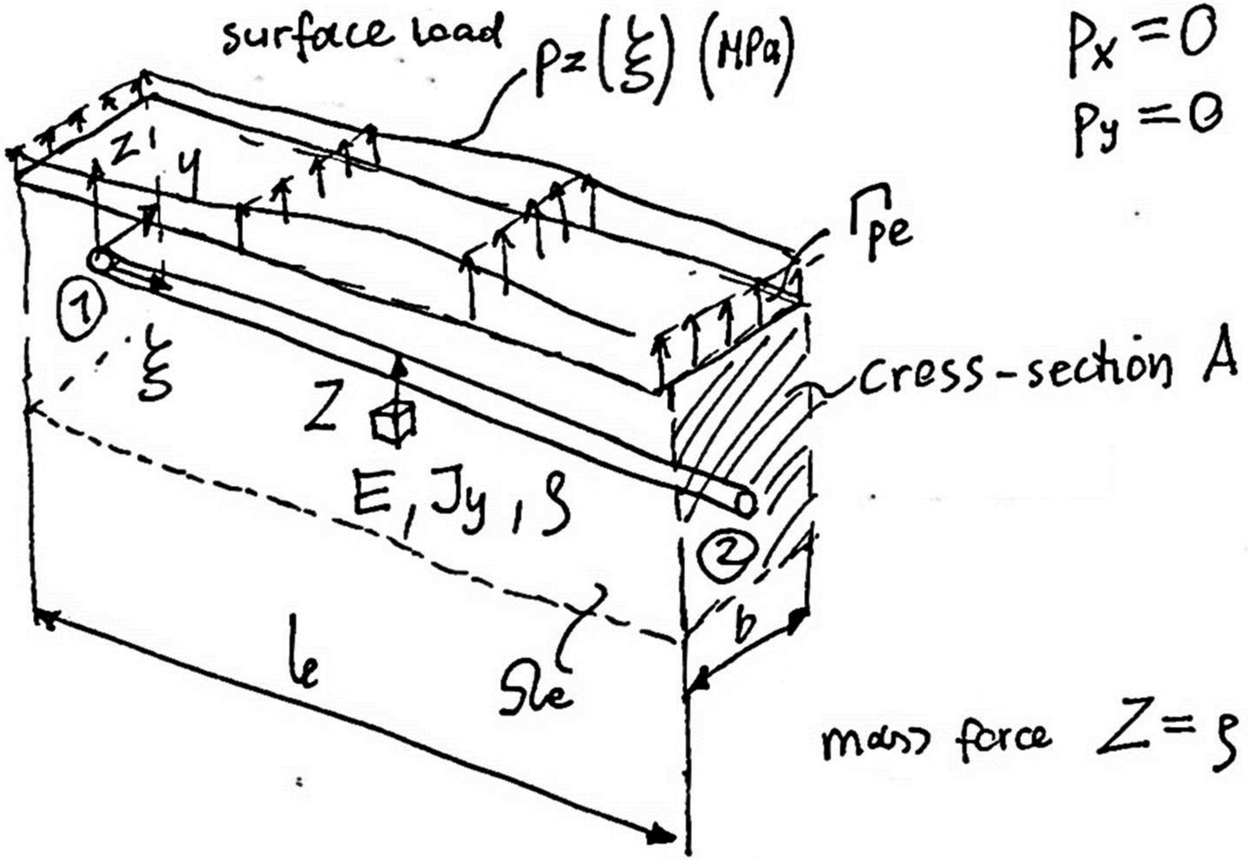
$$U_e = \frac{1}{2} \underset{1\times 4}{\lfloor q \rfloor_e} \cdot EJ_y \cdot \begin{bmatrix} \int_0^{l_e} N_1'' N_1'' d\xi & \int_0^{l_e} N_1'' N_2'' d\xi & \int_0^{l_e} N_1'' N_3'' d\xi & \int_0^{l_e} N_1'' N_4'' d\xi \\ \int_0^{l_e} N_2'' N_1'' d\xi & \int_0^{l_e} N_2'' N_2'' d\xi & \int_0^{l_e} N_2'' N_3'' d\xi & \int_0^{l_e} N_2'' N_4'' d\xi \\ \int_0^{l_e} N_3'' N_1'' d\xi & \int_0^{l_e} N_3'' N_2'' d\xi & \int_0^{l_e} N_3'' N_3'' d\xi & \int_0^{l_e} N_3'' N_4'' d\xi \\ \int_0^{l_e} N_4'' N_1'' d\xi & \int_0^{l_e} N_4'' N_2'' d\xi & \int_0^{l_e} N_4'' N_3'' d\xi & \int_0^{l_e} N_4'' N_4'' d\xi \end{bmatrix} \cdot \underset{4\times 1}{\{q\}_e} =$$

$$= \frac{1}{2} \underset{1\times 4}{\lfloor q \rfloor_e} \cdot \underset{4\times 4}{[k]_e} \cdot \underset{4\times 1}{\{q\}_e}$$

local stiffness matrix of a beam element

$$\underset{4\times 4}{[k]_e} = \frac{2EJ_y}{l_e^3} \begin{bmatrix} 6 & 3l_e & -6 & 3l_e \\ 3l_e & 2l_e^2 & -3l_e & l_e^2 \\ -6 & -3l_e & 6 & -3l_e \\ 3l_e & l_e^2 & -3l_e & 2l_e^2 \end{bmatrix}$$

potential energy of loading:



$p_x = 0$

$p_y = 0$

mass force $Z = \varrho \cdot a_z(\xi)$ $\left(\frac{N}{mm^3}\right)$

$$W_e = \int_{\Omega_e} \underset{1\times3}{\lfloor X \rfloor} \underset{3\times1}{\{u\}} d\Omega_e + \int_{\Gamma_{pe}} \underset{1\times3}{\lfloor P \rfloor} \cdot \underset{3\times1}{\{u\}} d\Gamma_{pe} =$$

$$= \int_{\Omega_e} \lfloor 0, 0, \rho a_z \rfloor \cdot \begin{Bmatrix} u \\ v \\ w \end{Bmatrix} d\Omega_e + \int_{\Gamma_{pe}} \lfloor 0, 0, p_z \rfloor \cdot \begin{Bmatrix} u \\ v \\ w \end{Bmatrix} d\Gamma_{pe} =$$

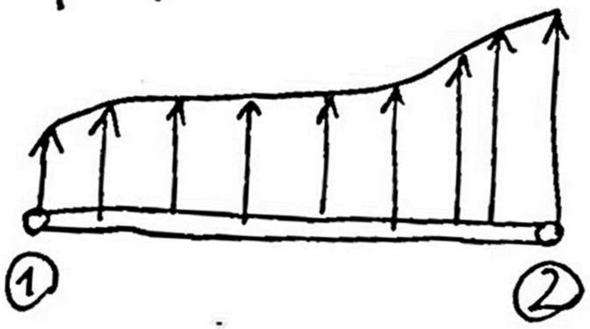
$$= \int_{\Omega_e} \rho a_z \cdot w \, d\Omega_e + \int_{\Gamma_{pe}} p_z \cdot w \, d\Gamma_{pe} =$$

$$= \int_0^{l_e} \rho a_z \cdot w \cdot \int_A dA \cdot d\xi + \int_0^{l_e} p_z \cdot w \int_{-\frac{b}{2}}^{\frac{b}{2}} dy \, d\xi =$$

$$= \int_0^{l_e} \rho a_z A \cdot w \, d\xi + \int_0^{l_e} p_z \cdot b \, w \, d\xi = \int_0^{l_e} \underset{\uparrow}{p(\xi)} \cdot w(\xi) \, d\xi$$

traction $p(\xi) = \rho a_z A + p_z \cdot b \quad \left(\frac{N}{mm}\right)$

traction $p(\xi)$

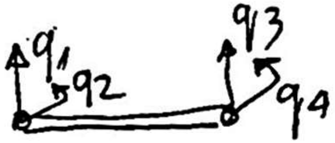


$$W_e = \int_0^{l_e} p(\xi) \cdot \underset{1\times 4}{\lfloor N \rfloor} \cdot \underset{4\times 1}{\{q\}_e} \, d\xi = \int_0^{l_e} p(\xi) \underset{1\times 4}{\lfloor N \rfloor} \, d\xi \cdot \{q\}_e =$$

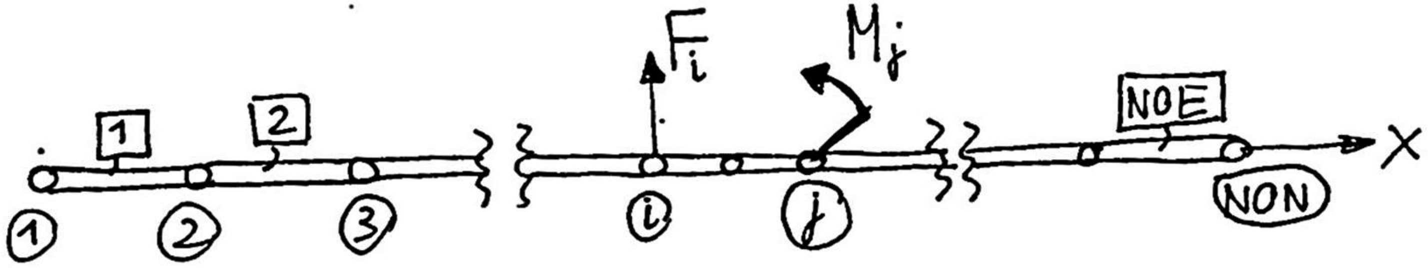
$$= \underset{\substack{\uparrow \\ \text{vecter cf equivalent load}}}{\lfloor F \rfloor_e} \cdot \{q\}_e$$

vecter cf equivalent load.

TOTAL POTENTIAL ENERGY OF A BEAM MODEL.



NODAL LOADS:



$$V = \sum_{e=1}^{NOE} U_e - \left(\sum_{e=1}^{NOE} W_e + \sum_i F_i w_i + \sum_j M_j \cdot \frac{dw}{dx}\Big|_j \right)$$

$$\frac{\partial V}{\underset{NDOF \times 1}{\partial \{q\}}} = 0 \Rightarrow \underset{NDOF \times NDOF}{[K]} \times \underset{NDOF \times 1}{\{q\}} = \underset{NDOF \times 1}{\{F\}}$$

$$\Rightarrow \underset{N \times N}{[K]} \cdot \underset{N \times 1}{\{q\}} = \underset{N \times 1}{\{F\}}$$

+ BOUNDARY CONDITIONS (NOF)

$N = NDOF - NOF$

DOF Solution

$$\underset{N\times 1}{\{q\}} = \underset{N\times N}{[K]^{-1}} \cdot \underset{N\times 1}{\{F\}}$$

Element solution

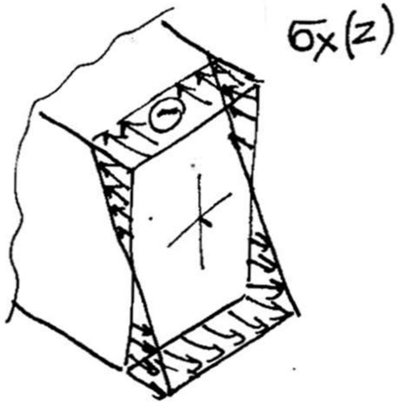
$$M_y(\xi) = EJ_y \underset{1\times 4}{\lfloor N''(\xi)\rfloor} \cdot \underset{4\times 1}{\{q\}_e} \,,\quad T_z(\xi) = -EJ_y \underset{1\times 4}{\lfloor N'''(\xi)\rfloor} \cdot \underset{4\times 1}{\{q\}_e}$$

$$\sigma_x(\xi, z) = -M_y(\xi)\cdot\frac{z}{J_y} = -E\underset{1\times 4}{\lfloor N''(\xi)\rfloor}\cdot\underset{4\times 1}{\{q\}_e}\cdot z$$

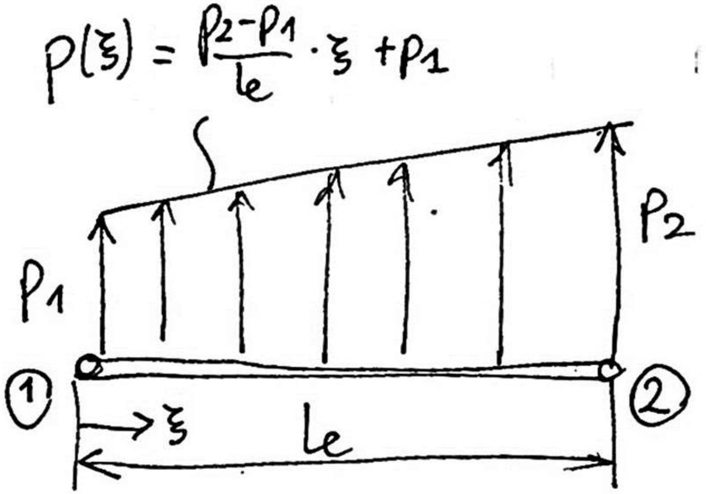
$$\tau_{xz}(\xi, z) = f(z)\cdot T_z(\xi) \; \frac{\text{rectangle}}{b\times h} \; \frac{3}{2}\left(1-\left(\frac{2z}{h}\right)^2\right)/bh\cdot T_z(\xi)$$

$$\varepsilon_x(\xi, z) = \sigma_x(\xi, z)/E = -\underset{1\times 4}{\lfloor N''\rfloor}\cdot\underset{4\times 1}{\{q\}_e}\cdot z$$

$$w(\xi) = \underset{1\times 4}{\lfloor N\rfloor}\cdot\underset{4\times 1}{\{q\}_e}$$



EXAMPLE. FIND COMPONENTS OF EQUIVALENT LOAD VECTOR FOR LINEARLY DISTRIBUTED TRACTION.



$$\lfloor F \rfloor_e = \lfloor F_{1e}, F_{2e}, F_{3e}, F_{4e} \rfloor_e$$

$$F_{1e} = \int_0^{l_e} p(\xi)\cdot N_1(\xi)\, d\xi = \int_0^{l_e} \left(\frac{p_2-p_1}{l_e}\cdot\xi + p_1\right)\left(1 - \frac{3}{l_e^2}\xi^2 + \frac{2}{l_e^3}\xi^3\right) d\xi =$$

$$= \frac{p_1 l_e}{2} + \frac{3}{20}(p_2-p_1)\cdot l_e$$

$$F_{2e} = \int_0^{l_e} p(\xi)\cdot N_2(\xi)\, d\xi = \int_0^{l_e} \left(\frac{p_2-p_1}{L_e}\cdot\xi + p_1\right)\left(\xi - \frac{2}{l_e}\xi^2 + \frac{1}{l_e^2}\xi^3\right) d\xi =$$

$$= \frac{p_1 l_e^2}{12} + \frac{1}{30}(p_2-p_1) l_e^2$$

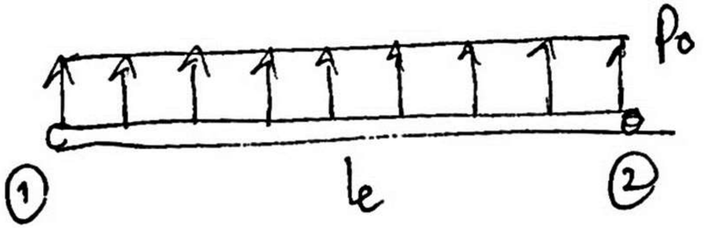
$$F_{3e} = \int_0^{l_e} p(\xi)\, N_3(\xi)\, d\xi = \int_0^{l_e} \left(\frac{p_2-p_1}{l_e}\cdot\xi + p_1\right)\left(\frac{3}{l_e^2}\xi^2 - \frac{2}{l_e^3}\xi^3\right) d\xi =$$

$$= \frac{p_1 l_e}{2} + \frac{7}{20}(p_2-p_1) l_e$$

$$F_{4e} = \int_0^{l_e} p(\xi)\cdot N_4(\xi)\, d\xi = \int_0^{l_e} \left(\frac{p_2-p_1}{l_e}\cdot\xi + p_1\right)\left(-\frac{1}{l_e}\xi^2 + \frac{1}{l_e^2}\xi^3\right) d\xi =$$

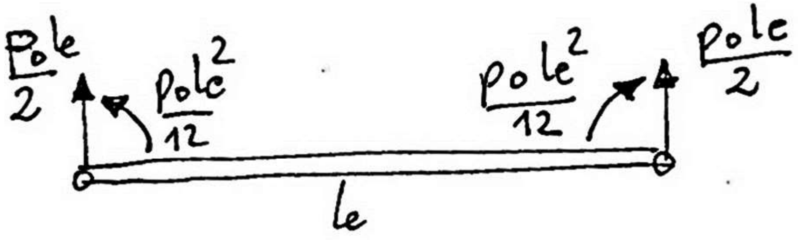
$$= -\frac{p_1 l_e^2}{12} - \frac{(p_2-p_1) l_e^2}{20}$$

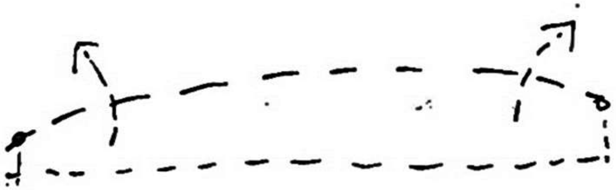
EXAMPLE. FIND $\lfloor F \rfloor_e$ FOR UNIFORMLY DISTRIBUTED TRACTION:



$p_1 = p_2 = p_0 = \text{const}$

$$\lfloor F \rfloor_e = \left\lfloor \frac{p_0 l_e}{2},\ \frac{p_0 l_e^2}{12},\ \frac{p_0 l_e}{2},\ -\frac{p_0 l_e^2}{12} \right\rfloor$$

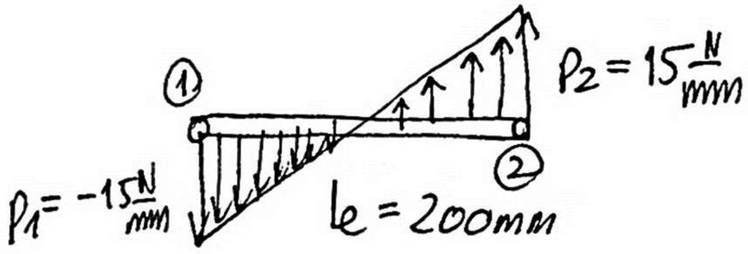


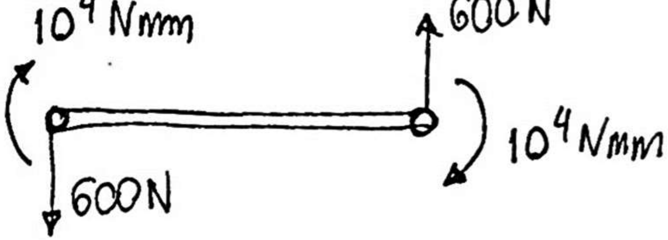


The load is kinematically equivalent when:

wirtual work of $\lfloor F \rfloor_{e}$ (1×4) = wirtual work of traction $p(\xi)$

EXAMPLE. FIND COMPONENTS OF $\lfloor F \rfloor_e$





$$F_{1e} = \frac{-15\frac{N}{mm}\cdot 200mm}{2} + \frac{3}{20}\left(15\frac{N}{mm} - \left(-15\frac{N}{mm}\right)\right)\cdot 200mm = -600N$$

$$F_{2e} = \frac{-15\frac{N}{mm}\cdot 200^2mm^2}{12} + \frac{1}{30}\left(30\frac{N}{mm}\right)\cdot 200^2mm^2 = -10^4 Nmm$$

$$F_{3e} = \frac{-15\frac{N}{mm}\cdot 200mm}{2} + \frac{7}{20}\left(30\frac{N}{mm}\right)\cdot 200mm = 600N$$

$$F_{4e} = -\frac{\left(-15\frac{N}{mm}\right)\cdot 200^2mm^2}{12} - \frac{\left(30\frac{N}{mm}\right)}{20}\cdot 200^2mm^2 = -10^4 Nmm$$